Prelude to the Giza Pyramids :

The Geometry of the Blunt Pyramid of Dahshur

By John A. R. Legon

Part 1 – The External Design

Through the severe simplicity of the design and in the perfection of the fine white limestone masonry, the Blunt Pyramid of Dahshur[1] exemplifies the minimalistic style of architecture so typical of the monuments which Egyptologists ascribe to the Fourth Dynasty of ancient Egypt, c.2550 BC. Nowhere in the construction of the passages and chambers was any ornamental detail allowed to interrupt the sequence of geometrical elements which define the dimensions. This austere architecture finds a parallel in the contemporary Bent Pyramid of Dahshur and in the Meydum Pyramid, but stands in dramatic contrast to the expansive, elaborate and ornate style of the Step Pyramid complex of Djoser at Sakkara, which is thought to have been built only 60 years previously. Indeed, the pyramids at Dahshur and Meydum are so far removed from the royal funerary edifices of the Third Dynasty in their internal arrangements and in the layout of the external cult buildings, that it seems reasonable to question whether the cultural, chronological, and religious relationships between these two distinct types of monument have been correctly ascertained.

Thanks to the survey carried out by Josef Dorner,[2] it is possible to show that the architect of the Blunt Pyramid developed the dimensions using geometrical concepts identical to those already traced by the present writer in the Bent Pyramid of Dahshur and in the Giza pyramids. Fundamental to this geometry is the use of the diagonals of squares and rectangles, the lengths of which are defined by Pythagoras' theorem for right triangles. The pyramid architects must have been familiar with the practical result of this theorem, regardless of whether they could supply a formal proof, and they must have been able to calculate square roots.

The calculation of square roots

Because the values of square roots are encountered in the geometry of the Blunt Pyramid, we need to dispel the common misconception that the calculation of square roots (other than for numbers which are perfect squares) was too difficult for the scribe to perform.[3] The awkward Egyptian notation using unit fractions could have been avoided by introducing appropriate powers of ten into the calculation, so that to find a first approximation for the square root of 2, for example, the scribe would have looked for the integer which when multiplied by itself came closest to giving 200, rather than the fractional number which when squared. came close to 2. As shown below for $\sqrt{2}$ and $\sqrt{3}$, increasingly accurate approximations could have been obtained by a process of trial and error.

1 Also known as the Northern Stone or Red Pyramid.

2 Josef Dorner, 'Neue Messungen an der Roten Pyramide', in *Stationen: Beiträge zur Kulturgeschichte Ägyptens* (eds. H. Guksch and D. Polz), Mainz, 1998.

3 The Berlin, Kahun and Moscow fragments of mathematical papyri all give the square roots of perfect squares,.

√2	√200	√20,000	√2,000,000
1 × 1 = 1	14 × 14 = 196	141 × 141 = 19881	1414 × 1414 = 1999396
2 × 2 = 4	15 × 15 = 225	142 × 142 = 20164	1415 × 1415 = 2002225
1 < √2 < 2	14 < √ 200 < 15	141 < √20,000 < 142	1414 < √2,000,000 < 1415
√3	√300	√30,000	√30,000,000
1 × 1 = 1	17 × 17 = 289	173 × 173 = 29929	1732 × 1732 = 2999824
2 × 2 = 4	18 × 18 =324	174 × 174 = 30276	1733 × 1733 = 3003289
1 < √3 < 2	17 < √300 < 18	173 < √30,000 < 174	1732 < √3,000,000 < 1733

These calculations were well within the capacity of the the Egyptian scribe to perform, and actually give the dimensions of the enclosing rectangle of the Giza site plan.[4]

Alternatively, approximations may be obtained by the empirical measurement of the sides of right triangles constructed on a suitable scale. Figure 1 shows the development of a 1,√3,2 right triangle from a double square or 1:2 rectangle, by marking off the length of 2 cubits as the hypotenuse to the width of one cubit as the base. Reference to the division of the cubit into seven palms (blue lines) shows that result of dividing 3 by 12/7 is slightly greater than √3 :

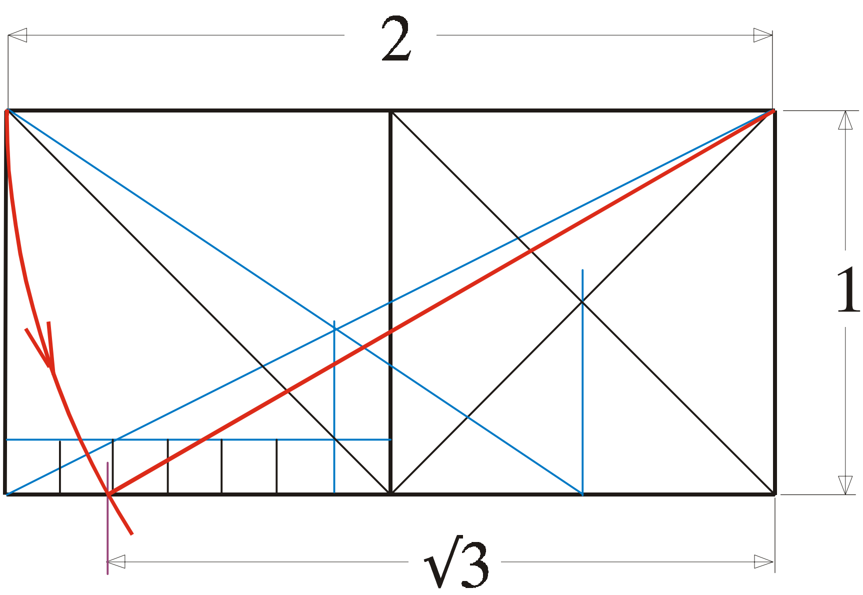

Figure 1.

The result of 3 / (12 / 7) being 7/4, we can obtain a more accurate approximation to √3 as the arithmetic mean of 12/7 and 7/4 :

$$(12/7 + 7/4) / 2 = 97 / 56 = 1.73214\ldots$$
$$\sqrt{3} = 1.73205\ldots$$

This excellent approximation to the square root of three is signalled in the Giza site plan by the division of the axial north-south distance of 1530 cubits from the north side of the Great Pyramid to the north side of the Third Pyramid, into line segments of 560 and 970 cubits.[5]

[4] Actually 1417.5 × 1732 cubits.

[5] See Legon, J.A.R., The Giza Site Plam Revisisted, *GM* 124 (1991), 69-78.

The addition of the nominal side of the Third Pyramid[6] then gives the overall north-south dimension of (202+1530) = 1732 = 1000√3 cubits

Similarly, the length of the diagonal in a square with a side of one cubit or seven palms is found by about direct measurement to be about10 palms, and so gives an initial approximation to √2 of 10/7. Dividing 2 by 10/7 then gives the complementary value of 5/7, with an arithmetic mean of:

$$(10/7 + 7/5)/2 = 99/70 = 1.1414285\ldots$$
$$\sqrt{2} = 1.4142135\ldots.$$

The external proportions of the Blunt Pyramid

For many years, most theories concerning the external proportions of the Blunt Pyramid were based on the measurements taken in 1839 by the British civil engineer John Shae Perring, who had been engaged in the construction of a railroad in the quarries at Turah. According to this early pioneer of pyramid exploration, the sides of the Blunt Pyramid were inclined at an angle of 43° 36' 11" to the horizontal, and had roughly the same slope as the upper portion of the Bent Pyramid.[7] Rising from a square base with a side-length of 719 feet 5 inches, according to Perring, the Blunt Pyramid would have been about equal in height to the Bent Pyramid, so that the profiles of these pyramids could be partially superimposed..

When expressed in terms of the Egyptian royal cubit of 20.620 inches or 0.52375 metres,[8] Perring's result for the side-length corresponds to 418.67 royal cubits. This has suggested an intended dimension of 420 cubits, which together with the fact that the inclination of the sides as reported by Perring corresponds to exactly 20 rise on 21 base,[9] has been taken as evidence that the design of the Blunt Pyramid was based upon the Pythagorean 20,21,29 right triangle. For the semibase of 210 cubits, the height of the pyramid would have been just 200 cubits, while the slant height or apothem would have been exactly 290 cubits.

In 1887, however, Flinders Petrie determined the face-angle by sighting down the sides of the core-masonry with a theodolite, and found that the angle was about one degree steeper than Perring had estimated:[10]

Table 1. Face-angle of Blunt Pyramid, Petrie:

North	East	South	West	Mean
44° 42'	44° 32'	44° 30'	44° 41	44° 36'

The inaccuracy of Perring's result was confirmed when Rainer Stadelmann reported an angle of about 45° following his discovery of some well-preserved fine white limestone casing still *in situ* near the centre of the pyramid's east side.[11] Perring had evidently measured the slope as 20 parts rise for 21 parts base using a straight-edge and spirit level, and then converted this

[6] Including the corner displacement of 0.5 cubit due to the rotation of the base with respect to the azimuth of the plan.

7 See H. Vyse and J.S.Perring, *Appendix to Operations carried on at the Pyramids of Gizeh in 1837,* Vol. III, (London,1840), 65. Legon, J.A .R. The Geometry of the Bent Pyramid, *Göttinger Miszellen* 116 (1990), 65-72.

8 For the Egyptian royal cubit of 20.620 inches or 0.52375 metres as determined by W.M.F. Petrie, *The Pyramids and Temples of Gizeh* (London,1883), p. 179

9 The measures of 218.5 m by 221.5 m as given by G.A. Reisner, *The Development of the Egyptian Tomb down to the Accession of Cheops* (Cambridge, Mass., 1935), 153, were not obtained by survey but measured off a plan of the pyramid.

[10] Petrie, A *Season in Egypt,* 27.

11 R.Stadelmann, *Die ägyptishen Pyramiden*, (Mainz am Rhein, 1991), 101.

proportion to an exact angle using tangent tables.

In 1997, Stadelmann invited Josef Dorner to undertake a new survey of the Blunt Pyramid which would include the substantial remains of the casing on the east side. Surface variations, which seem to show that the casing-stones had been cut with the required slope before being laid, gave a range in the observed casing-angle from 44° 17' to 45° 04', with the following overall result:

Mean casing-angle of Blunt Pyramid, Dorner = 44° 44'

Dorner concluded that the intended slope was 1 rise on 1 base, giving a *seked* of 7 palms,[12] and a casing-angle equal to the 45° diagonal of a square. Petrie, however, contended that the angle was clearly not 45° and that the only rule likely to have been employed by the builders was 7 slope on 5 base, with a theoretical angle of 44° 24' 55".[13] A gradient of 10 run on 7 rise is also possible, for an angle of 44° 25' 37".

The Dimensions of Height and Base

Owing to the destruction of the casing at the corners and the encumbrance of the sides with mounds of debris, the exact side-length of the Blunt Pyramid has proved difficult to ascertain, and Dorner's measurement refers specifically to the south side. There is, however, no reason to doubt that the base was laid out as a more or less perfect square with four equal sides.

Side-Length of Blunt Pyramid, Dorner = 219.08 metres = 418.3 Giza cubits

The difference with respect to Perring's finding is just 20 cm.

Taking a length of royal cubit about 0.3 mm longer than the Giza royal cubit of 0.52375 m, Dorner concluded that the builders had set the length of the sides to the whole number of 418 cubits, but without offering any explanation for this choice of dimension.

We can now calculate the height of the Blunt Pyramid from the measured mean face-angle of 44° 44' and the semi-base of 219.08 / 2 or 109.54 metres:[14]

Height of Blunt Pyramid = 109. 54 × tan(44° 44') = 108.53 m = 207.2 cubits[15]

As illustrated in figure 2, these dimensions are the fundamental products of a geometrical development starting from a square with a side of just 1000 cubits. The square is placed in a circle, the diameter of which is equal to the diagonal of the square, or 1000√2 cubits. The sides of the square are also chords to the circle, and cut off four equal segments, each with a height given by the difference between the radius of the circle and half the side-length of the square, that is:

$$500.(\sqrt{2} - 1) = 207.10\ldots \text{ cubits}$$

Thus the height of the Blunt Pyramid is defined by the most elementary geometry.

12 As used in the Rhind mathematical papyrus of the Hyksos era. There is no evidence to show that the *seked* dates back to the Old Kingdom.

13 Petrie, *A Season in Egypt*, 1887 (London 1888), 27.

14 Dorner assumes a casing-angle of 45° and so gives a height equal to half the base.

15 The slope of the reconstructed pyramidion may also be taken into account.

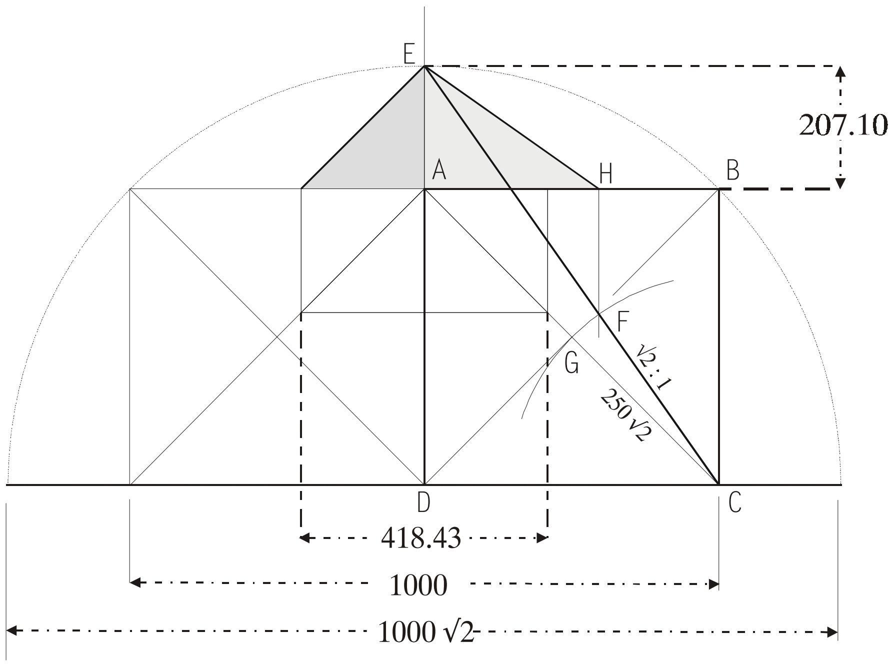

Fig.2

The diagonal of the base of the Blunt Pyramid, and hence the side-length of the base, can now be constructed as shown in figure 2, by first drawing the line EC with a slope of $\sqrt{2}:1$. Next marking off along this line from C to F, the semi-diagonal of the square ABCD with a length of $250\sqrt{2}$ cubits, the semi-diagonal of the base is obtained by dropping a perpendicular from F onto the pyramid base-line at H. The line CF represents the hypotenuse of a 1, $\sqrt{2}$, $\sqrt{3}$ right triangle, and corresponds to the mean lower slope of the Bent Pyramid. [16] We have:

Semi-diagonal of base of Blunt Pyramid, $AH = 250.(2 - \sqrt{2} / \sqrt{3}) = 295.87$ cubits

Side-length of Blunt Pyramid $= 2 \times AH / \sqrt{2} = 500.(\sqrt{2} - 1 / \sqrt{3}) = 418.43$ cubits

Hence the survey data is in perfect agreement with the theoretical side-length

Since the height of the Blunt Pyramid is now defined as $500.(\sqrt{2} - 1)$ or 207.10… cubits, the theoretical slope of the arris-edges will be $500.(\sqrt{2} - 1)$ rise on $250.(2 - \sqrt{2} / \sqrt{3})$ base, which reduces to:

Slope of arris-edges $= \sqrt{6}.(\sqrt{2} - 1) / (\sqrt{6} - 1) = 0.69997866 \sim 7$ rise on 10 base

Thus the slope is the inverse of the profile of 10 rise on 7 base, which was used for the lower slope on the north side of the Bent Pyramid and in other places near the base, instead of the geometrically simple profile of $\sqrt{2}$ rise on 1 base.[17]

It is now evident that the $\sqrt{2}:1$ profile provided an intermediate stage in the construction of the 10:7 profile, and was possibly retained as the overall mean lower casing-angle for this reason.

Now taking the 7:10 profile for the arris-edges of the Blunt Pyramid together with the approximation to $\sqrt{2}$ of 99/70, the theoretical face-angle will be:

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[16] Following Lauer and based upon Petrie's survey data.

[17] See Legon, 'The Problem of the Bent Pyramid', *GM* 132, 1992, 49-56.

Casing angle of Blunt Pyramid = atan (7 / 10 × 99 / 70) = atan (99 / 100) = 44° 42’ 44”

Thus once again we have an almost perfect agreement with the survey data. We may surmise that the numbers 7 and 10 had some special significance or symbolism extending beyond the fact that the decimal system was used for counting while the cubit was curiously divided into seven palms.

At the same time, however, the face angle of 99 slope on 100 base is so close to the 45° diagonal of a square that the 1:1 profile may be considered valid to a first approximation. In this case, the height of the pyramid will be equal to the semi-base, while the slant height or apothem will represent the hypotenuse of a 1,1,√2 right triangle. As shown in Figure.3, each face of the pyramid then contains two right triangles whose base, height and hypotenuse are in the ratio of 1:√2:√3. If we now take the base of each triangle to be equal to the whole number semibase of the pyramid of 418/2 or 209 cubits, then the corner or arris-edges will measure 209√3 cubits, with the following interesting result:

Length of arris-edges of Blunt Pyramid = 209 × √3 = 361.996... ~ 362 cubits

The arris-edges are thus almost exactly equal in length to side-length of the Bent Pyramid.of 362 cubits, and the ratio of 362:209 between the sides of the Bent Pyramid and the semi-base of the Blunt Pyramid is an excellent approximation to the square root of three.:

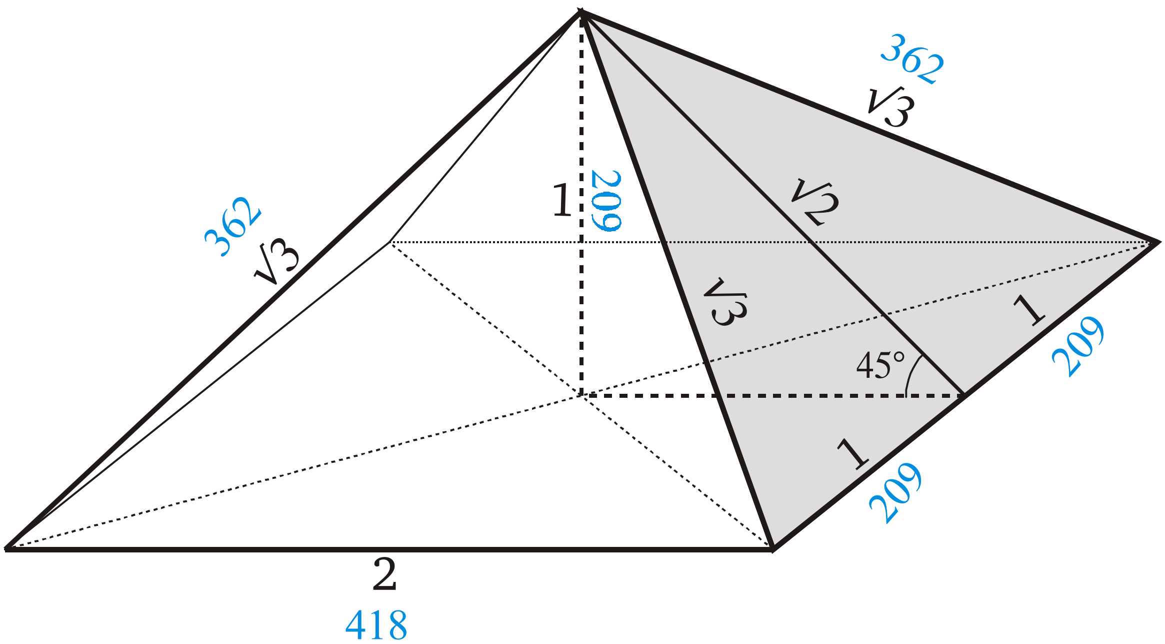

Figure 3

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This is one of a series of ‘best’ approximations to √3, each being unsurpassed for accuracy by any other fraction with a smaller denominator and numerator. Other examples are 97 / 56 and 265 / 153 , which are linked because 56 + 97 = 153 and 3 × 56 + 97 = 265. Likewise, we find that 153 + 265 = 418 while 3 × 153 + 265 = 2 × 362 Far from being arbitrary, therefore, the use of these numbers in succesive pyramids suggests that the architect was well acquainted with the theory of numbers and left a record in the pyramids he designed.

Still more surprisingly, just as the side-length of the Bent Pyramid became the arris-edge length of the Blunt Pyramid, so also we find that the side-length of the Blunt Pyramid became the arris-edge length of the Great Pyramid. Given the height of 280 cubits and side of 140 pi cubits, this last dimension is theoretically 418.47 cubits, or only 0.04 cubit more than the ideal non-integer side-length of the Blunt Pyramid as defined above.

At Giza, the arris-edge length of the Great Pyramid and side length of the Blunt Pyramid are equal to the distance westwards from the Great Pyramid to the centre of the Second Pyramid, of (213 + 411/2) or 418.5 cubits. The arris-edge length of the Second Pyramid is just 400 cubits, or twice the round-figure side-length of the Third Pyramid.[18]

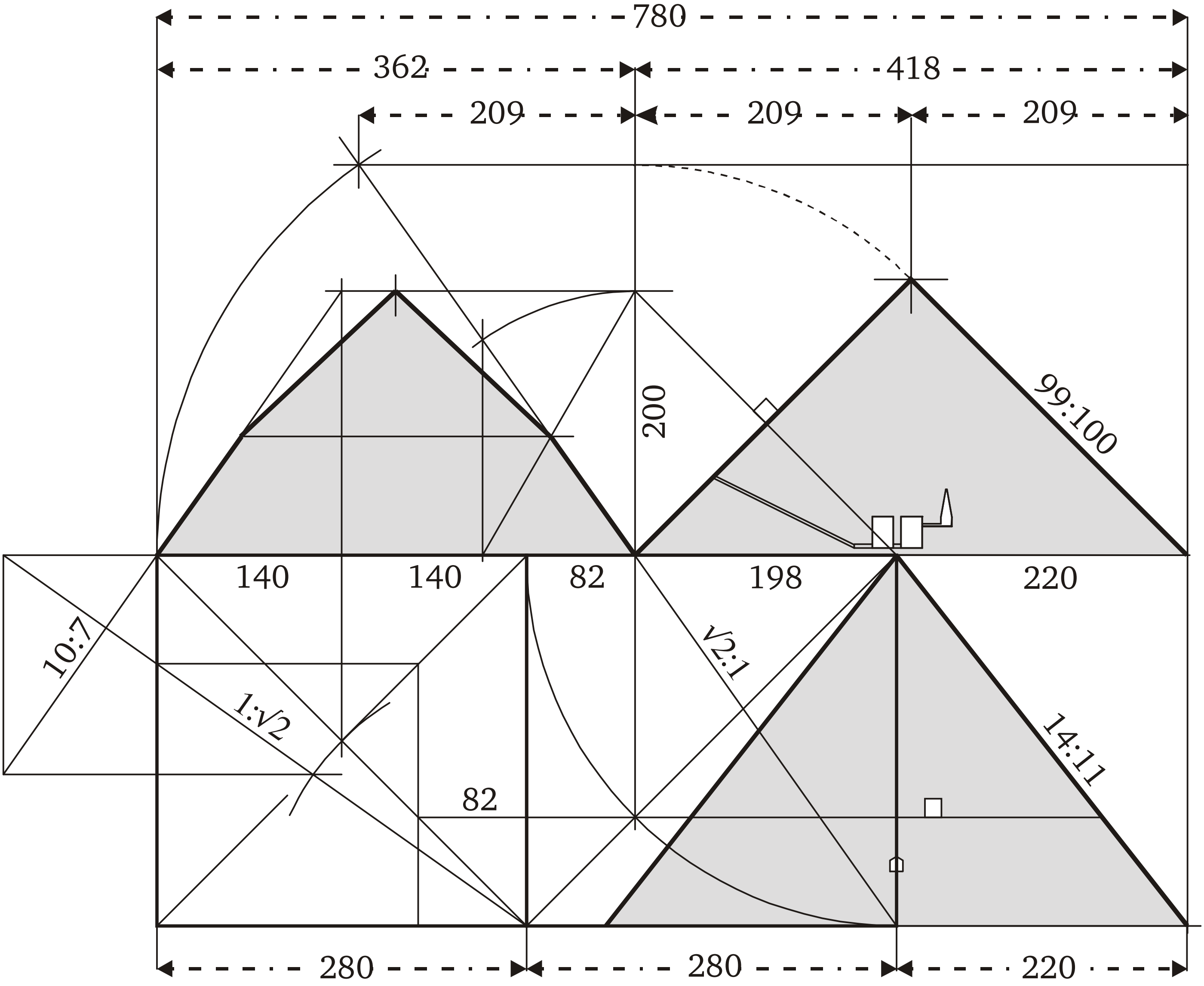

Fig.4

Finally, as shown in fig. 4, the relationships which exist between the Bent and the Blunt Pyramids of Dahshur and the Great Pyramid of Giza can be illustrated by means of a coherent geometrical scheme in which the virtually exact profiles of all three pyramids are developed side-by-side using pure straight-edge and compass geometry. The dimensions are generated in proportion to an initial unit with the value of 280 cubits, which defines the height of the Great Pyramid and plausibly represents the royal cubit of 28 fingers as the starting point for the dimensonal scheme.

18 Using Petrie's measurements of the height and base, the length of the arris-edges of the Second Pyramid can be calculated as 8246 +/- 15 inches or 399.9 +/- 0.7 cubits

The development begins with one quadrant of a circle which is constructed on the height of the Great Pyramid, and has a radius of 280 cubits which is also the side of a square. The arc of the circle intersects the diagonal of the square at the level of the King's Chamber, which is 280 / √2 or 198 cubits below the apex of the pyramid, and thus (280 – 198) or 82 cubits above the base, as shown by Petrie's survey data. Adding this element horizontally to the height of the pyramid then gives the base of the Bent Pyramid, which is now seen to be (280 + 82) or 362 cubits.

Since the mean lower slope of the Bent Pyramid corresponds to √2 rise on 1 base, it is defined by the dimensions of 280 and 198 cubits as shown in figure 4. On the north side of the Bent Pyramid, however, the lower slope was set to the profile of 10 rise on 7 base, and this slope and can now be constructed with great accuracy using the geometry described for the arris-edges of the Blunt Pyramid. Projecting this slope of 10:7 over the semi-width of a square of 280 cubits gives a vertical dimension of 200 cubits, and so defines the height of the Bent Pyramid. This is divided in the ratio of √2 : √3 through the intersection of a line ascending at the slope of √2 rise on 1 base, with a line descending from the level of the apex with a slope of √3 rise on 1 base. The division of the height places the bend in the casing at the level of 89.9 cubits above the base.

The semi-base of the Blunt Pyramid is now constructed by applying the side-length of the Bent Pyramid of 362 cubits to the casing-line with the profile of √2 rise on 1 base. This is the hypotenuse of a 1,√2,√3 right triangle and defines a base side of. 362 / √3 or 209 cubits, which is doubled to give the integer side-length of the Blunt Pyramid of 418 cubits. Adding this to the side-length of the Bent Pyramid then gives a base-line of (362 + 418) or 780 cubits which extends to (780 – 560) or 220 cubits beyond the centre of the Great Pyramid, and so defines the pi-proportion of the pyramid in proportion to the height of 280 cubits. Lastly, the slope of the Blunt Pyramid of 99 rise on 100 base is obtained by constructing a perpendicular to the profile of 200 rise on 198 base, which is already defined by the height of the Bent Pyramid of 200 cubits in proportion to the upper height of the Great Pyramid of 198 cubits.

Part Two : The Internal Design of the Blunt Pyramid – to follow.

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